O369. Let $a, b, c > 0$. Prove that

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq \sqrt{3(a^2+b^2+c^2)}.$$

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Let $p := ab+bc+ca, q := abc$ and let $a+b+c=1$(due to homogeneity).

Then

$$\sum_{cyc}\frac{a^2+bc}{b+c}=\frac{1}{p-q}\sum_{cyc}\left(a^2+bc\right)(1-b)(1-c)=\frac{1}{p-q}\sum_{cyc}\left(a^2+bc\right)(a+bc)=$$

$$\frac{1}{p-q}\left(a^3+b^3+c^3+3abc+abc\,(a+b+c)+a^2b^2+b^2c^2+c^2a^2\right)=$$

$$\frac{1+3q-3p+3q+q+p^2-2q}{p-q}=\frac{1+5q-3p+p^2}{p-q}$$

and inequality becomes

$$\frac{1-3p+p^2+5q}{p-q} \geq \sqrt{3(1-2p)}.$$

Also note that $0<p\leq\dfrac{1}{3}$ (because $ab+bc+ca\leq\dfrac{(a+b+c)^2}{3}$) and $q\geq\dfrac{(1-p)(4p-1)}{6}$

(Schure Inequality $\sum_{cyc}a^2(a-b)(a-c)\geq 0$ in 1-p-q notation).

Since $\dfrac{1-3p+p^2+5q}{p-q}$ increasing in $q>0$ and $q\geq\max\left\{0,\dfrac{(1-p)(4p-1)}{6}\right\}$
then

$$\frac{1-3p+p^2+5q}{p-q}\geq\frac{1-3p+p^2+5\cdot\dfrac{(1-p)(4p-1)}{6}}{p-\dfrac{(1-p)(4p-1)}{6}}=\frac{1+7p-14p^2}{4p^2+p+1}$$

and suffices to prove inequalities:

$$\frac{1+7p-14p^2}{4p^2+p+1}\geq\sqrt{3(1-2p)} \iff \left(1+7p-14p^2\right)^2\geq 3(1-2p)\left(4p^2+p+1\right)^2$$

for $p\in[1/4,1/3]$

and

$$\frac{1-3p+p^2}{p}\geq\sqrt{3(1-2p)} \text{ for } p\in(0,1/4].$$